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Metric operators for quasi-Hermitian Hamiltonians and symmetries of equivalent Hermitian Hamiltonians

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Abstract
We give a simple proof of the fact that every diagonalizable operator that has a real spectrum is quasi-Hermitian and show how the metric operators associated with a quasi-Hermitian Hamiltonian are related to the symmetry generators of an equivalent Hermitian Hamiltonian.

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1. Introduction

Given a separable Hilbert space $\mathcal{H}$ and a linear operator $H : \mathcal{H} \to \mathcal{H}$ that has a real spectrum and a complete set of eigenvectors, one can construct a new (physical) Hilbert space $\mathcal{H}_{\rm phys}$ in which H acts as a self-adjoint operator. This allows for the formulation of a consistent quantum theory where the observables and in particular Hamiltonian need not be self-adjoint with respect to the standard (L^2-) inner product on $\mathcal{H}$ [1]. The physical Hilbert space $\mathcal{H}_{\rm phys}$ and the observables are determined in terms of a (bounded, everywhere-defined, invertible) positive-definite metric operator $\eta_+ : \mathcal{H} \to \mathcal{H}$ that renders H pseudo-Hermitian [2], i.e., H satisfies[1]

$$H^\dagger = \eta_+ H \eta_+^{-1}. \tag{1}$$

This marks the basic significance of the metric operator η_+. The positivity of η_+ implies that H belongs to a special class of pseudo-Hermitian operators called quasi-Hermitian operators [3].

The fact that for a given linear operator H with a real (discrete) spectrum and a complete set of eigenvectors, one can always find a (positive-definite) metric operator η_+ fulfilling (1) has been established in [4], and the role of antilinear symmetries such as $\mathcal{PT}$ has been elucidated

[1] Here and throughout this article, we use $A^\dagger$ to denote the adjoint of a linear operator A that is defined using the inner product $\langle\cdot|\cdot\rangle$ of $\mathcal{H}$ according to: $\langle\psi|A\phi\rangle = \langle A\psi|\phi\rangle$ for all $\psi, \phi \in \mathcal{H}$.

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in [5][2]. Further investigation into the properties of η_+ has revealed its non-uniqueness [3, 7, 9] and the unitary equivalence of H and the Hermitian Hamiltonian

$$h := \rho H \rho^{-1}, \tag{2}$$

where $\rho := \sqrt{\eta_+}$, [10].[3] The latter observation has been instrumental in providing an objective assessment of the 'complex ($\mathcal{PT}$-symmetric) extension of quantum mechanics' [11, 12]. It has also played a central role in clarifying the mysteries associated with the wrong-sign quartic potential [13]. In short, the pseudo-Hermitian quantum theory that is defined by the Hilbert space $\mathcal{H}_{\text{phys}}$ and the Hamiltonian H admits an equivalent Hermitian description in terms of the (standard) Hilbert space $\mathcal{H}$ and the Hermitian Hamiltonian h. However, the specific form of h depends on the choice of η_+. This has motivated the search for alternative methods of computing the most general metric operator for a given H, [14–19].

In this paper we first give a simple proof of the existence of metric operators η_+ and then relate η_+ to the symmetries of an equivalent Hermitian Hamiltonian.

2. The existence of metric operators

Let $H : \mathcal{H} \to \mathcal{H}$ be a (closed) operator with a real spectrum, and suppose that it is diagonalizable, i.e., there are operators $T, H_d : \mathcal{H} \to \mathcal{H}$ such that T is invertible (bounded and hence closed),

$$H = T^{-1} H_d T, \tag{3}$$

and H_d is diagonal in some orthonormal basis of $\mathcal{H}$. The latter property implies that H_d is a normal operator. Furthermore, because H and H_d are isospectral, the spectrum of H_d is also real. This together with the fact that H_d is normal imply that it is Hermitian (self-adjoint).

Next, recall that because T is a closed, invertible operator it admits a polar decomposition [20]:

$$T = U\rho, \tag{4}$$

where U is a unitary operator and $\rho = |T| := \sqrt{T^\dagger T}$ is invertible and positive (definite). Inserting (4) into (3) and introducing

$$h := U^\dagger H_d U, \tag{5}$$

we find

$$H = \rho^{-1} h \rho. \tag{6}$$

Because ρ is positive definite, so is $\eta_+ := \rho^2$. Because H_d is Hermitian and U is unitary, h is Hermitian. In view of this and the fact that ρ is also Hermitian, (6) implies $H^\dagger = \eta_+ H \eta_+^{-1}$. This proves the existence of a metric operator η_+ that makes H, η_+-pseudo-Hermitian.

The above proof is shorter than the one given in [4]. But it has the disadvantage that it does not offer a method of computing η_+.

3. Metric operators and symmetry generators

Let η_+ and η'_+ be a pair of metric operators rendering H pseudo-Hermitian, $\rho := \sqrt{\eta_+}$, and $\rho' := \sqrt{\eta'_+}$. Then the Hermitian Hamiltonian operators

$$h := \rho H \rho^{-1}, \qquad h' := \rho' H \rho'^{-1} \tag{7}$$

[2] The alternative approach using the so-called $\mathcal{CPT}$-inner product [6] is equivalent to a specific choice of the metric operator [7, 8].

[3] Given a positive operator $X : \mathcal{H} \to \mathcal{H}$, $\sqrt{X}$ denotes the unique positive square root of X.

are unitary equivalent to H, [10]. It is easy to see that h and h' are related by the similarity transformation

$$h' = AhA^{-1}, \tag{8}$$

where

$$A := \rho'\rho^{-1}. \tag{9}$$

Now, taking the adjoint of both sides of (8) and using the fact that h and h' are Hermitian, we find

$$[A^\dagger A, h] = 0. \tag{10}$$

This means that $A^\dagger A$ is a (positive-definite) symmetry generator for the Hamiltonian h. Furthermore, (9) and $\eta'_+ = \rho'^2$ lead to the curious relation

$$\eta'_+ = \rho\, A^\dagger A\, \rho. \tag{11}$$

Another immediate consequence of (9) is

$$A^\dagger = \rho^{-1}A\rho, \tag{12}$$

i.e., A is ρ^{-1}-pseudo-Hermitian [2].

It is easy to show that the converse relationship also holds, i.e., given an invertible linear operator $A : \mathcal{H} \to \mathcal{H}$ that satisfies (9) and (12), the operator η' defined by

$$\eta'_+ := \rho\, A^\dagger A\, \rho. \tag{13}$$

renders H, η'-pseudo-Hermitian.

The above analysis shows that given a metric operator $\eta_+ = \rho^2$ for the Hamiltonian H, we can express any other metric operator for H in the form

$$\eta'_+ = \rho\, S\, \rho, \tag{14}$$

where S is a positive-definite symmetry generator of h such that there is a ρ^{-1}-pseudo-Hermitian operator A satisfying

$$S = A^\dagger A. \tag{15}$$

In practice, the construction of the symmetry generators S of the Hermitian operator h is easier than that of the ρ^{-1}-pseudo-Hermitian operators A. This calls for a closer look at the structure of A.

In view of (15), we can express A in the form

$$A = U\,\sigma, \tag{16}$$

where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator and $\sigma := \sqrt{S}$. This reduces the characterization of A to that of appropriate unitary operators U that ensure ρ^{-1}-pseudo-Hermiticity of A.

Inserting (16) in (12) and introducing

$$B := \rho\, U, \tag{17}$$

we find

$$B^\dagger = \sigma\, B\sigma^{-1}. \tag{18}$$

That is, B is σ-pseudo-Hermitian. Moreover, (17) implies

$$\eta_+ = BB^\dagger. \tag{19}$$

Conversely, given a positive-definite symmetry generator S and a $\sqrt{S}$-pseudo-Hermitian operator B satisfying (19), we can easily show that the operator

$$U := \rho^{-1}B \tag{20}$$

is unitary and A given by (16) is ρ^{-1}-pseudo-Hermitian. As a result, the most general metric operator η'_+ is given by (14), alternatively

$$\eta'_+ = (\sqrt{S}\rho)^\dagger(\sqrt{S}\rho), \tag{21}$$

where S is a positive-definite symmetry generator of h such that there is a $\sqrt{S}$-pseudo-Hermitian operator B satisfying $\eta_+ = BB^\dagger$.

4. Concluding remarks

The existence of a positive-definite metric operator η_+ that renders a diagonalizable Hamiltonian operator H with a real spectrum η_+-pseudo-Hermitian can be directly established using the well-known polar decomposition of closed operators. Previously, we have pointed out that one can describe the most general η_+ in terms of a given metric operator and certain symmetry generators A of H, [7]. Here we offer another description of the most general η_+ in terms of certain positive-definite symmetry generators S of a given equivalent Hamiltonian h. Unlike the symmetry generators A of H that are non-Hermitian, the operators S are standard Hermitian symmetry generators. This makes the results of this paper more appealing.

For the cases that h is an element of a dynamical Lie algebra $\mathcal{G}$ with $\mathcal{H}$ furnishing a unitary irreducible representation of $\mathcal{G}$, one can identify the positive-definite symmetry generators S with certain functions of a set of mutually commuting elements of $\mathcal{G}$ that includes h. For example, one can construct S for the two-level system, where $\mathcal{G} = u(2)$, or the generalized (and simple) Harmonic oscillator where $\mathcal{G} = su(1, 1)$, [21]. These respectively correspond to the general two-level quasi-Hermitian Hamiltonians [18] and the class of quasi-Hermitian Hamiltonians that are linear combinations of x^2, p^2 and $\{x, p\}$ such as the one considered in [22]. For these systems one can also construct a metric operator η_+ and its positive square root ρ. Nevertheless, the implementation of formula (14) for obtaining the most general metric operator proves impractical. This is because it is not easy to characterize the general form of $\sqrt{S}$-pseudo-Hermitian operators B that would fulfil $\eta_+ = BB^\dagger$.

Although formula (14) seems to be of limited practical value, it is conceptually appealing because it traces the non-uniqueness of the metric operator to the symmetries of the equivalent Hermitian Hamiltonians.

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